EJERCICIO RESUELTO

Resuelve, utilizando un cambio de variable, las siguientes integrales:

a) $\int x^2 e^{x^3}\,dx$ **b)** $\int x\sqrt{x-1}\,dx$ **c)** $\int \frac{dx}{\sqrt{e^x-1}}$

RESOLUCIÓN

a) En este caso, el cambio de variable lo podríamos haber realizado mentalmente; en las páginas anteriores ya hemos resuelto directamente integrales de este tipo. Veamos cómo calcularla mediante un cambio de variable:

Hallamos $t = x^3 \rightarrow dt = 3x^2\,dx$; queda así:

$$\int x^2 e^{x^3} dx = \frac{1}{3}\int e^{x^3} \cdot \underbrace{3x^2\,dx}_{dt} = \frac{1}{3}\int e^t dt = \frac{1}{3}e^t + k = \frac{1}{3}e^{x^3} + k$$

b) Cambio: $t = \sqrt{x-1} \rightarrow t^2 = x-1 \rightarrow x = t^2+1 \rightarrow dx = 2t\,dt$

$$\int x\sqrt{x-1}\,dx = \int (t^2+1)\cdot t \cdot 2t\,dt = \int (2t^4 + 2t^2)\,dt = \frac{2t^5}{5} + \frac{2t^3}{3} + k =$$

$$= \frac{2\sqrt{(x-1)^5}}{5} + \frac{2\sqrt{(x-1)^3}}{3} + k$$

c) Cambio: $t = \sqrt{e^x-1} \rightarrow t^2 = e^x - 1 \rightarrow e^x = t^2+1 \rightarrow x = ln(t^2+1) \rightarrow dx = \frac{2t}{t^2+1}dt$

$$\int \frac{dx}{\sqrt{e^x-1}} = \int \frac{2t}{(t^2+1)t}\,dt = \int \frac{2t}{t^2+1}\,dt = 2\,arc\,tg\,t + k = 2\,arc\,tg\,\sqrt{e^x-1} + k$$

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1 Resuelve por sustitución las siguientes integrales:

a) $\int \frac{dx}{1+\sqrt[3]{x}}$ (Cambio: $t = \sqrt[3]{x}$)

b) $\int \frac{1+x}{1+\sqrt{x}}\,dx$

c) $\int \frac{x\,dx}{\sqrt{x+2}}$ (Cambio: $t = \sqrt{x+2}$)

d) $\int \frac{dx}{x\sqrt{2x-1}}$

e) $\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$ (Cambio: $t = \sqrt{e^x+1}$)

f) $\int \frac{e^{2x}\,dx}{e^x+1}$

g) $\int \frac{dx}{x\sqrt{x^2-1}}$ (Cambio: $t = \frac{1}{x}$)

h) $\int \sqrt{1-x^2}\,dx$ (Cambio: $x = sen\,t$)

Utiliza la fórmula: $cos^2\,t = \frac{1}{2} + \frac{cos\,2t}{2}$

1 a) $t = \sqrt[3]{x} \rightarrow x = t^3 \rightarrow dx = 3t\,dt$

$$\int \frac{dx}{1+\sqrt[3]{x}} = \int \frac{3t\,dt}{1+t} = \int \left(3 + \frac{-3}{t+1}\right) dt =$$

$$= 3t - 3\,ln|t+1| + k = 3\sqrt[3]{x} - 3\,ln|\sqrt[3]{x} + 1| + k$$

b) $t = \sqrt{x} \rightarrow x = t^2 \rightarrow dx = 2t\,dt$

$$\int \frac{1+x}{1+\sqrt{x}}\,dx = \int \frac{2t + 2t^3}{1+t}\,dt =$$

$$= \int \left(2t^2 - 2t + 4 + \frac{-4}{t+1}\right) dt =$$

$$= \frac{2t^3}{3} - t^2 + 4t - 4\,ln|1+t| + k =$$

$$= \frac{2\sqrt{x^3}}{3} - x + 4\sqrt{x} - 4\,ln|1 + \sqrt{x}| + k$$

c) $t = \sqrt{x+2} \rightarrow t^2 = x + 2 \rightarrow$

$\rightarrow x = t^2 - 2 \rightarrow dx = 2t\,dt$

$$\int \frac{x\,dx}{\sqrt{x+2}} = \int \frac{(t^2-2)\cdot 2t\,dt}{t} = \int (2t^2 - 4)\,dt =$$

$$= \frac{2t^3}{3} - 4t + k = \frac{2\sqrt{(x+2)^3}}{3} - 4\sqrt{x+2} + k$$

d) $t = \sqrt{2x-1} \rightarrow t^2 = 2x - 1 \rightarrow x = \dfrac{t^2+1}{2} \rightarrow$

$\rightarrow dx = t\,dt$

$$\int \frac{dx}{x\sqrt{2x-1}} = \int \frac{t\,dt}{\frac{t^2+1}{2}\cdot t} = \int \frac{2\,dt}{t^2+1} =$$

$$= 2arc\,tg\,t + k = 2\,arc\,tg\,(\sqrt{2x-1}) + k$$

e) $t = \sqrt{e^x+1} \rightarrow t^2 = e^x + 1 \rightarrow e^x = t^2 - 1 \rightarrow$

$\rightarrow x = ln|t^2 - 1| \rightarrow dx = \dfrac{2t}{t^2-1}\,dt$

$$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \int (2t^2 - 2)\,dt = \frac{2t^3}{3} - 2t + k =$$

$$= \frac{2\sqrt{(e^x+1)^3}}{3} - 2\sqrt{e^x+1} + k$$

f) $t = e^x \rightarrow x = ln\,t \rightarrow dx = \dfrac{1}{t}\,dt$

$$\int \frac{e^{2x}dx}{e^x+1} = \int \frac{t\,dt}{t+1} = \int \left(1 + \frac{-1}{t+1}\right) dt =$$

$$= t - ln\ |t + 1| + k = e^x - ln\ |e^x + 1| + k$$

g) $t = \dfrac{1}{x} \rightarrow x = \dfrac{1}{t} \rightarrow dx = \dfrac{-1}{t^2}\ dt$

$$\int \frac{dx}{x\sqrt{x^2 - 1}} = \int \frac{-dt}{\sqrt{1 - t^2}} = arc\ cos\ t + k =$$

$$= arc\ cos\ \frac{1}{x} + k$$

h) $x = sen\ t \rightarrow dx = cos\ t\ dt$

$$\int \sqrt{1 - x^2}\ dx = \int \sqrt{1 - sen^2\ t} \cdot cos\ t\ dt =$$

$$\int cos^2\ t\ dt = \int \left(\frac{1}{2} + \frac{cos\ 2t}{2}\right) dt = \frac{t}{2} + \frac{sen\ 2t}{4} + k =$$

$$= \frac{t}{2} + \frac{2\ sen\ t\ cos\ t}{4} + k = \frac{arc\ sen\ x}{2} + \frac{x\sqrt{1 - x^2}}{2} + k$$